Mathematics-III

Solved Exam Paper 2019

Q1 a) If y=A cos (log x) + B sin (log x), Show that x^2y_{n+2} + (2n+1) * y_{n+1} + (n^2 + 1) y_n=0 where y_n= d^ny/dx^n

If y = Acos(logx) + Bsin(logx) ------------(1)

Differentiating (1) w.r.t x, we get

y_1 = -asin(logx)1/x + bcos(logx)1/x

xy_1 = -asin(logx) + bcos(logx) ------------(2)

Diff 2 again w.r.t x, then we get

$xy_2 + y_1$ = -acos(logx)1/x - bsin(logx)1/x

$x^2y_2 + xy_1$ = -[acos(logx) + bsin(logx)]

$x^2y_2 + xy_1 = -y$

$y_2x^2 + y_1x + y = 0$ ----------------------------(3)

Diff 3 by Leibnitzle theorem n times, we get

$[\, y_{n+2}x^2 + nc1\, y_{n+1}2x + nc2\, y_n.2 \,] + [\, y_{n+1}x + nc1\, y_{n-1}] + y_n = 0$

$x^2y_{n+2} + 2nxy_{n+1} + xy_{n+1} + n(n-1)y_n + ny_n + y_n = 0$

$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Q2 a) Show that the following function is continuous at the point (0,0):

$$f(x,y) \begin{cases} \frac{2x^3+3y^3}{x^2+y^2} & , (x,y) \;!= (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Answer:

$$f(x,y) = \begin{cases} \frac{2x^3+3y^3}{x^2+y^2} & , \ (x,y) \ != (0,0) \\ 0 & , \ (x,y) = (0,0) \end{cases}$$

$$0 <= \left| \frac{2x^3 + 3y^3}{x^2 + y^2} \right| <= \frac{2|x^3|}{x^2 + y^2} + \frac{3|y^3|}{x^2 + y^2} = 0$$

$$0 <= \frac{2|x^3|}{x^2} + \frac{3|y^3|}{y^2} = 2|x| + 3|y|$$

$$lt_{(x,y)\to(0,0)} \ 2|x| + 3|y| = 0$$

∴ by the squeez theorem, we conclude that

$lt_{(x,y)\to(0,0)} f(x,y) = (0,0)$ that is => $lt_{(x,y)\to(0,0)} \frac{2x^3 + 3y^3}{x^2 + y^2} = 0$

∴ f is continous at (0,0)

Q2 b) If $z(x+y) = x^2 + y^2$ show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

solution:

$z(x+y) = x^2 + y^2$

$z = \frac{x^2+y^2}{x+y}$

$\frac{\partial z}{\partial x} = \frac{(x+y)2x-(x^2+y^2).1}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$

$\frac{\partial z}{\partial y} = \frac{(x+y)2y-(x^2+y^2).1}{(x+y)^2} = \frac{-x^2+2xy+y^2}{(x+y)^2}$

$(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}) = \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{-x^2+2xy+y^2}{(x+y)^2} = \frac{2(x-y)}{(x+y)}$

$(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y})^2 = \frac{4(x-y)^2}{(x+y)^2}$ -- -1

$4(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}) = 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{-x^2+2xy+y^2}{(x+y)^2}\right]$

$= \frac{4(x-y)^2}{(x+y)^2}$ -- -----2

From 1 & 2

$(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y})^2 = 4(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y})$

Q3 a)Transform the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0$$ **into polar coordinates**.

Solution:

we have $x = r\cos\theta$, $y = r\sin\theta$

$r^2 = x^2 + y^2$, $\theta = \tan^{-1}y/x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} . \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} . \frac{\partial \theta}{\partial x} \Rightarrow \frac{\partial u}{\partial x}(\cos\theta) - \frac{\partial u \sin\theta}{\partial\theta \; r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r} . \frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta \partial u}{r \; \partial\theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial x}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\left(\frac{\partial u}{\partial\theta}\right)\right) - \frac{\sin\theta}{r} . \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\right)$$

$$= \cos\theta\left(\cos\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta \partial u}{r^2 \; \partial\theta} - \frac{\sin\theta}{r} . \frac{\partial^2 u}{\partial r \partial\theta}\right) - \frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial u}{\partial r} + \right.$$

$$\left. \cos\theta\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\cos\theta \partial u}{r \; \partial\theta} - \frac{\sin\theta \partial^2 u}{r \; \partial\theta^2}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta\cos\theta \partial u}{r^2 \quad \partial\theta} - \frac{\sin\theta\cos\theta \; \partial^2 u}{r \quad \partial r\partial\theta} + \frac{\sin^2\theta \partial u}{r \; \partial r} -$$

$\frac{\sin\theta\cos\theta}{r}$ $\frac{\partial^2 u}{\partial\theta\partial u}$ + $\frac{\sin\theta\cos\theta\partial u}{r^2\ \partial\theta}$ + $\frac{\sin^2\theta\partial^2 u}{r^2\ \partial\theta^2}$

= $\cos^2\theta\frac{\partial^2 u}{\partial r^2}$ + 2$\frac{\sin\theta\cos\theta\partial u}{r^2\ \partial\theta}$ - 2 $\frac{\sin\theta\cos\theta\ \partial^2 u}{r\ \partial r\partial\theta}$ + $\frac{\sin^2\theta\partial u}{r\ \partial r}$ +

$\frac{\sin^2\theta\partial^2 u}{r^2\ \partial\theta^2}$ ---(1)

= $\frac{\partial u}{\partial y}$ = $\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$ + $\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y}$

= $\frac{\partial u}{\partial r}$ y/r + $\frac{\partial u}{\partial\theta}\frac{x}{(x^2+y^2)}$

= $\frac{\partial u}{\partial r}\sin\theta$ + $\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}$

= $\frac{\partial^2 u}{\partial y^2}$ = $\frac{\partial}{\partial y}(\frac{\partial u}{\partial y})$

= $(\sin\theta\ \frac{\partial}{\partial u} + \frac{\cos\theta\ \partial}{r\ \partial\theta})(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta\partial u}{r\ \partial\theta})$

= $\sin\theta\frac{\partial}{\partial r}(\sin\theta\ \frac{\partial u}{\partial r} + \frac{\cos\theta\partial u}{r\ \partial\theta})$ + $\frac{\cos\theta\ \partial}{r\ \partial\theta}(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta\partial u}{r\ \partial\theta})$

= $\sin\theta[\ \sin\theta\frac{\partial^2 u}{\partial r^2} - \frac{\cos\theta\partial u}{r^2\ \partial\theta} + \frac{\cos\theta}{r} + \frac{\partial^2 u}{\partial r\partial\theta}]$ + $\frac{\cos\theta}{r}[\ \cos\theta\ [\frac{\partial u}{\partial r}$ +

$\sin\theta\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\sin\theta\partial u}{r\ \partial\theta} + \frac{\cos\theta\partial^2 u}{r\ \partial\theta^2}]$

= $\sin^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta\cos\theta\partial u}{r^2\ \partial\theta} + \frac{\sin\theta\cos\theta\ \partial^2 u}{r\ \partial r\partial\theta} + \frac{\cos^2\theta}{r}(\frac{\partial u}{\partial r}) + \frac{\sin\theta\cos\theta}{r}$

$(\frac{\partial^2 u}{\partial r \partial \theta})$ - $\frac{sin\theta cos\theta \partial u}{r^2 \quad \partial \theta}$+ $(\frac{\partial^2 u}{\partial \theta^2})$

= **$\sin^2\theta \frac{\partial^2 u}{\partial r^2}$**- $2\frac{sin\theta cos\theta \partial u}{r^2 \quad \partial\theta}$+$2\frac{sin\theta cos\theta \ \partial^2 u}{r \quad \partial r \partial \theta}$+ $\frac{cos^2\theta}{r}(\frac{\partial u}{\partial r})$ +

$(\frac{\partial^2 u}{\partial \theta^2})$ -- (2)

By adding 1 & 2

$\frac{\partial^2 u}{\partial x^2}$+ $\frac{\partial^2 u}{\partial y^2}$= $(\sin^2\theta+\cos^2\theta)\frac{\partial^2 u}{\partial r^2}$+$(\sin^2\theta+\cos^2\theta)1/r\frac{\partial u}{\partial r}$+ $(\sin^2\theta+\cos^2\theta)1/r^2\frac{\partial^2 u}{\partial \theta^2}$

= $(\frac{\partial^2 u}{\partial r^2}$+ $1/r\frac{\partial u}{\partial r}$+$1/r^2\frac{\partial^2 u}{\partial \theta^2})$ ans

Q4 Find the extreme values of f(x,y ,z) = 2x+3y+z, such that $x^2+y^2=5$ and x+z=1

f(x,y,z) = 2x + 3y + z ----------------------(1)

f(x,y) = $(x^2 + y^2)$ - 5 -----------------(2)

y(x,z) = x+z-1 ------------------(3)

Lagranges Multipliers Equations are

$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} + m \frac{\partial \Psi}{\partial x} = 0$

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} + m \frac{\partial \Psi}{\partial y} = 0$

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} + m \frac{\partial \Psi}{\partial z} = 0$

$2 + \lambda(2x) + m(1) = 0$ ------------(4)

$3 + \lambda(2y) + m(0) = 0$ ------------(5)

$1 + \lambda(0) + m(1) = 0$ ------------(6) => $m = -1$

putting the values of m in (4) and (5), we get

$2 + 2\lambda x - 1 = 0 \Rightarrow 2\lambda x = -1, \quad x = -\frac{1}{2\lambda}$

$3 + 2\lambda y = 0 \Rightarrow 2\lambda y = -3, \quad y = -\frac{3}{2\lambda}$

putting the values of x,y in $x^2 + y^2 = 5$, we get

$\frac{1}{4\lambda^2} + \frac{9}{4\lambda^2} = 5 \Rightarrow \frac{10}{4\lambda^2} = 5$

$2\lambda^2 = 1 \Rightarrow \lambda = \pm \frac{1}{\sqrt{2}}$

we know that , $x = -\frac{1}{2\lambda} = \pm \frac{\sqrt{2}}{2} = \pm \frac{1}{\sqrt{2}}$

$y = -\frac{3}{2\lambda} = \pm \frac{3\sqrt{2}}{2} \pm \frac{3}{\sqrt{2}}$

From (3) , x+z =1 or z = 1-x

$z = 1 + - \frac{1}{\sqrt{2}}$

putting $x = \frac{1}{\sqrt{2}}$, $y = \frac{3}{\sqrt{2}}$ and $z = 1 - \frac{1}{\sqrt{2}}$ in equation (1), we get

$f = \frac{2}{\sqrt{2}} + \frac{9}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}} = \frac{10}{\sqrt{2}} + 1$ = 5Ö2 + 1

putting $x = -\frac{1}{\sqrt{2}}$, $y = -\frac{3}{\sqrt{2}}$ and $z = 1 + \frac{1}{\sqrt{2}}$ in equation (1), we get

$f = 2\frac{1}{\sqrt{2}} + 3\left(\frac{-3}{\sqrt{2}}\right) + \left(1 + \frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} - \frac{9}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}$

f = 1 - 6 Ö2 ans

Q5 Evaluate òò$_s$ F.n.ds where F = 4xz $\hat{i}$ + $y^2\hat{j}$ + yz$\hat{k}$ and S is surface of the cube bounded by x=0, y=1, z=0 , z=1, by using Gauss divergence theorem

s.no	surface	ds		
1	OABC	-k	dxdy	z=
2	DEFG	k	dxdy	z:

3	OAFG	-ĵ	dxdz	y:
4	BCDE	ĵ	dxdz	y:
5	ABEF	i^	dydz	x
6	OCDG	-i^	dydz	x:

òò$_s$ F¯ n^ ds = òò$_{OABC}$ F¯ n^ ds + òò$_{DEFG}$ F¯ n^ ds + òò$_{OAFG}$ F¯ n^ ds + òò$_{BCDE}$ F¯ n^ ds + òò$_{ABEF}$ F¯ n^ ds +

òò$_{OCDG}$ F¯ n^ ds ---------------------------------
(1)

òò$_{OABC}$ F¯ n^ ds = òò$_{OABC}$ (4xz i^ + y^{2}ĵ + yzk^)(-k)dxdy

$$= \int_0^1 \int_0^1 -yz\,dx\,dy = 0 \text{(as } z = 0)$$

òò$_{DEFG}$ (4xz i^ + y^2ĵ + yzk^)(k)dxdy = òò$_{DEFG}$ yzdxdy

$\int_0^1\int_0^1 y(1)dxdy = \int_0^1 dx\left[\frac{y^2}{2}\right]_{0-1}$ = $[x]_{0\text{-}1}$ = 1/2

òò$_{OAFG}$ (4xz i^ + y^2ĵ + yzk^)(-j^)dxdz = òò$_{OAFG}$ y^2dxdz = 6 (as y =0)

òò$_{BCDE}$(4xz i^ + y^2ĵ + yzk^)(j^)dxdz = òò$_{BCDE}$ ($-y^2$)dxdz

$-\int_0^1 dx\int_0^1 dz$ = $(x)_{0\text{-}1}(z)_{0\text{-}1}$ = -1 (as y = 1)

òò$_{ABEF}$ (4xz i^ + y^2ĵ + yzk^).i^ dydz = òò4xzdydz

$= \int_0^1\int_0^1 4(1)zdydz$ => $4(y)_{0\text{-}1}\left(\frac{z^2}{2}\right)_{0-1}$ = $4\ (1)\left(\frac{1}{2}\right)$ = 2

òò$_{OCDG}$(4xz i^ + y^2ĵ + yzk^)(-i^) dydz = $\int_0^1\int_0^1 4(1)zdydz$ = 0 (as x = 0)

on putting these values in (1), we get

òò$_s$ F¯ n^ ds = 0 + 1/2 + 6 -1 + 2 + 0 => 3/2

Q 6 (a) Evaluate $\frac{\partial}{\partial\theta}${A*{ B * C }}

i^ ĵ k^

(B*C) = cosq -sinq -3

2 3 -1

= i^ (sinq + 9) – ĵ (-cosq + 6) + k^ (3cosq + 2sinq)

i^ ĵ k^

A*(B * C) = sinq cosq Q

(sinq + 9) (cosq - 6) (3cosq + 2sinq)

= i^[cosq(3cosq + 2sinq) - q(cosq – 6)]

- ĵ[sinq(3cosq + 2sinq) - q(sinq + 9)]

+ k^[sinq(cosq – 6) - cosq (sinq + 9)]

at q = 0

=> $\hat{i}$[cos0(3cos0 + 2sin0) - 0(cos0 – 6)]

- $\hat{j}$[sin0(3cos0 + 2sin0) - 0(sin0 + 9)]

- $\hat{j}$[sin0(3cos0 + 2sin0) - 0(sin0 + 9)]

=> 3$\hat{i}$ - 9$\hat{k}$ ans

Q 6 A particle moves along the curve $x=t^3 + 1$, $y=t^2$, $z= 2t+5$ where t is the time. Find the components of the velocity and acceleration at t=1 in the direction i+j+3k

$x = t^3 + 1$, $y = t^3$, $z = 2t + 5$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\bar{r} = (t^3 + 1)\hat{i} + (t^3)\hat{j} + (2t + 5)\hat{k}$

velocity = $\frac{dr}{dt}$ = $3t^2\,\hat{i} + 2t\hat{j} + 2\hat{k}$

when t = 1, we have , $\frac{dr}{dt}$= $3\,\hat{i} + 2\hat{j} + 2\hat{k}$

unit velocity along ($\hat{i} + \hat{j} + 3\hat{k}$) = ($\hat{i} + \hat{j} + 3\hat{k}$) / Ö (1 + 1 + 9)

$= \frac{1}{\sqrt{11}}(\hat{i} + \hat{j} + 3\hat{k})$

component of velocity $(3\hat{i} + 2\hat{j} + 2\hat{k})$ along $(\hat{i} + \hat{j} + 3\hat{k})$

$= (3\hat{i} + 2\hat{j} + 2\hat{k}) \cdot \frac{1}{\sqrt{11}}(\hat{i} + \hat{j} + 3\hat{k})$

$= \frac{1}{\sqrt{11}}(3+2+6) = \frac{11}{\sqrt{11}}$ **= Ö11 ans**

Q7 (a) Solve by the method of variation of parameters

$\frac{d^2y}{dx^2} + n^2y = \sec nx$

solution: - $(D^2 + n^2)y = \sec nx$

$m^2 + n^2 = 0 \Rightarrow m = \pm ni$

$cf = c_1\cos nx + c_2\sin nx$

$y = A'\cos nx + B'\sin nx$ -----------------(I)

by diff. Equation (1)

$(-nA'\sin nx + B'n\cos nx = \sec nx)$ -------------(II)

Now multiplying by nsinx in equation (I) & multiplying by cosx in equation (II)

nsinnx(A'cosnx + B'sinnx = 0)

cosnx(-nA'sinnx + B'ncosnx = secnx)

A' nsinnx cosnx + B' $nsin^2$ nx = 0

- A' nsinnx cosnx + B' $ncos^2$ nx = 1

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nB' (sin^2nx + cos^2nx) = 1

B' = $\frac{1}{n}$

$\frac{dB}{dx}$ = $\frac{1}{n}$

dB = $\frac{dx}{n}$

B = $\frac{x}{n}$ + C

A'cosnx + B'sinnx = 0

$A'\cos nx + \frac{1}{n}\sin nx \quad = 0$

$A' = -\frac{1 \sin nx}{n \cos nx} = -\frac{1}{n}\tan nx$

$\int dA = -\frac{1}{n}\int \tan nx\, dx + C_2$

$A = -\frac{1}{n^2}\log \sec nx + C_2$

$y = [-\frac{1}{n^2}\log \sec nx + C_2]\cos nx + (\frac{x}{n} + C_1)\sin nx$

$Y = C_1 \sin nx + C_2 \cos nx + \frac{x}{n} \sin nx - \frac{1}{n^2} \cos nx. \log \sec nx$

Q 7 (b) solve $\frac{d^2y}{dx^2}$ - 2 tanx $\frac{dy}{dx}$ + 5y = sec x. e^x

$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = \sin x. e^x$

$y'' - 2\tan xy' + 5y = \sin x. e^x$

compare with $y'' + 8y' + qy = R$

$p = -2\tan x$, $q = 6$, $R = \sin x. e^x$

for C.F $\mu = e^{-1/2\int p dx}$

$q_1 = q - \frac{1dp}{2dx} - \frac{p^2}{4}$

$R_1 = \frac{R}{4}$

$\mu = e^{-1/2\int -2\tan x \, dx} = e^{\log \sec x} = \sec x$

$q_1 = 5 - \frac{d}{dx}(-2\tan x) - \frac{4\tan^2 x}{4}$

$= 5 + \sec^2 x - \tan^2 x = 6$

$R_1 = \frac{\sin x e^x}{4}$

$\frac{d^2v}{dx^2} + q_1 v = R_1$

$(D^2 + 6)v = \frac{\sin x e^x}{4}$

A.E $= m^2 + 6 = 0 \Rightarrow m^2 = -6 \Rightarrow m = +- 6i$

CF $= (C_1 \cos 6x + C_2 \sin 6x)$

P.I $= \frac{1}{(D^2+6)} \frac{\sin x e^x}{4} \Rightarrow \frac{1}{4} \frac{1}{(D^2+6)} (\sin x . e^x)$

$= \frac{e^x}{4} \frac{1}{[(D+1)^2 + 6]} \sin x$

$= \frac{e^x}{4} \frac{1}{(D^2+1+2D+6)} \sin x \Rightarrow \frac{e^x}{4} \frac{1}{(-(1^2)+1+2D+6)} \sin x$

$= \frac{e^x}{4} \frac{1}{(-1+1+2D+6)} \sin x \Rightarrow \frac{e^x}{4} \frac{1}{(2D+6)} \sin x$

$= \frac{e^x}{8} \frac{D-3}{(D^2-9)} \sin x \Rightarrow \frac{e^x}{8} \frac{D-3}{(-1-9)} \sin x$

$= \frac{e^x}{-80}(D-3)\sin x \Rightarrow \frac{e^x}{-80}(D\sin x - 3 \sin x)$

$= \frac{e^x}{-80}(\cos x - 3 \sin x)$

C.S $= (C_1 \cos 6x + C_2 \sin 6x) - \frac{e^x}{80}(\cos x - 3 \sin x$