Aryabhatta Knowledge University (AKU)

Mechanical Engineering

Mathematics-III

Solved Exam Paper 2019

Instructions

i. The marks are indicated in the right-hand margin.

ii. There are **NINE** questions

iii. Attempt **FIVE** questions

iv. Questions no. 1 is compulsory

v. Relevant statistical data are given at the end of question paper.

1. Q1. Choose the correct answer (any seven):

a. If p_n is the legendre polynomial of first kind, then the value of $\int_{-1}^{1} p_n(x)dx$ is.

i. 0

ii. $\frac{2}{2n+1}$

iii. 2

iv. 1

b. If j_n is the bessel's function of first kind, then the value of $2j_n$, is

i. $j_{n-1} + j_{n+1}$

ii. $j_n - j_{n+1}$

iii. $j_n + j_{n+1}$

iv. $j_{n-1} - j_{n+1}$

c. The particular integral of $(D^2 - D'^2)Z = x - y$, is

i. $\frac{1}{2}x^3 + yx^2$

ii. $\frac{1}{3}x^3 - \frac{1}{2}yx^2$

iii. $\frac{1}{6}x^3 - \frac{1}{2}yx^2$

iv. $x^3 + \frac{1}{2}yx^2$

d. The function $x^3 + x + 1$ in terms of legendre polynominal is equal to

i. $p_3 + 5p_1 - 5p_0$

ii. $\frac{2}{5}p_3 + \frac{8}{5}p_1 + p_o$

iii. $\frac{2}{3}p_3 + p_2 + p_1 - p_0$

iv. $\frac{1}{5}p_3 + p_2 + 5p_1 - \frac{1}{5}p_0$

e. Let the joint probability density functions of the continuous random variable X and Y be.

$$f(x,y) = \begin{pmatrix} k(x^2 + y^2) & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{pmatrix}$$

Then the margin density of X is.

i. $3x^2 + 1$

ii. $\frac{3}{5}(2x^2 + 1)$

iii. $\frac{1}{2}(3x^2 + 1)$

iv. $\left(x^2 + \frac{2}{3}\right)$

f. If $P(A \cap B) = \frac{1}{4}, P(A \cup B) = \frac{3}{4}, P(\bar{A}) = \frac{2}{3}$ then $P\left(A/\bar{B}\right)$ is equal to

i. $\frac{1}{3}$

ii. $\frac{1}{4}$

iii. $\frac{1}{2}$

iv. $\frac{3}{8}$

g. Let A,B and C be any three mutually exclusive events. Which one of the following is incorrect?

i. $P(A \cap B \cap C) = P(A) + P(B) + P(C)$

ii. $P(A \cap B) = 0$

iii. $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

iv. $P(B \cap C) = 0$

h. If μ is the mean and σ is the standard deviation of a set of measurement which are normally distributed , then percentage of measurement within the range $\mu + 2\sigma$

i. 98

ii. 95.44

iii. 99.73

iv. 95

i. If the density function of gamma distribution is

$$f(x) = \begin{cases} \dfrac{\left(x^{\alpha-1} e^{\frac{-x}{\beta}}\right)}{\beta^{\alpha}\Gamma\alpha} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Then variation is equal to.

i. $A\beta$

ii. β

iii. $\alpha^2\beta$

iv. $\alpha\beta^2$

j. The moment generating function of a continuous random variable X be given as

$$M_X(t) = (1-t)^7 \text{for} |t| < 1$$

Then its mean and variance is

i. $\left(7, \frac{1}{7}\right)$

ii. $\left(\frac{1}{7}, \frac{1}{7}\right)$

iii. $\left(\frac{1}{7}, 7\right)$

iv. (7,7)

Q2. A) Solve

$$x(y^2+z)\frac{\partial z}{\partial x} - y(x^2+z)\frac{\partial z}{\partial y} = (x^2-y^2)z$$

Answer: $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$

Choose the multipliers as $\frac{1}{x'}\frac{1}{y'}\frac{1}{z}$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = k\{\frac{1}{k}x(y^2+z) + \frac{1}{y}(k^2+z) + \frac{1}{z}(x^2+z)$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = k(y^2+z-k^2-z+x^2-y^2)$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

On integration

$$\log x + \log y + \log z = \log c_1$$

$$\log(xyz) = \log c_1$$

$$xyz = c_1$$

now again choose the multipliers as x,y,-1

$$xdx - ydy - dz = k[x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)]$$

$$xdx - ydy - dz = k[x^2y^2 + x^2z - y^2x^2 - y^2z - zx^2 + zy^2]$$

$$xdx - ydy - dz = 0$$

on integration

$$\frac{x^2}{2} - \frac{y^2}{2} - z = c_2$$

$$x^2 - y^2 - 2z = 2c_2$$

Now on combining both eq. we get a general solution

$$f(\mathrm{xyz}, x^2 - y^2 - 2z) = 0$$

2. Q3. State and prove Rodrigues formula .

Sol- we derive a formula for the legendre polynomials

Formula

Now proof

Let

We shall first establish that the nth derivation of u, that is u_n is a so the legendre differentiate eq.

$$(1 - x^2)y^{``} - 2\mathrm{xy}^{`} + n(n+1)u = 0 - - - - - - 1$$

Differ w.r. to x

$$\frac{\mathrm{du}}{\mathrm{dx}} = u_1 = n(x^2 - 1)^{n-1}.2x$$

Or

$$(x^2 - 1)u_1 = 2\mathrm{nx}(x^2 - 1)^u$$

i.e $(x^2 - 1)u_1 = 2\mathrm{nxu}$

Diff. w.r. to x again, we have

$$(x^2-1)u_2+2xu_1=2x(xu_1+u)$$

Now differ. The result in timer by applying lebuitz theorem for derivation of a product given by

$$(\mathrm{uv})_n=\mathrm{uv}_n+\mathrm{nu}_1 v_{n-1}+\frac{n(n-1)}{2}u_2 v_{n-2}+\ldots u_n v$$

$$[(x^2-1)u_2]_n+2(xu_1)_n=2n(\mathrm{xv}_1)_n+2_x u_n$$

$$\left[(x^2-1)u_{n+2}+\mathrm{x}2*u_{x+1}+\frac{n(n-1)}{2}.2u_x\right]+2(xu_{x+1}+x\mathrm{cu})$$

$$(x^2-1)u_{n+2}+2n*u_{n+1}+(n^2-n)u_n+2*u_{n+1}+2_n u_x=2\mathrm{nx}u_{n+1}+2n^2u_x+2$$

$$(x^2-1)u_{n+2}+2xu_{n+1}-n^2u_n-nu_x=0$$

Or

$$(1-x^2)u_{n+2}-2*u_{n+1}+n(n+1)\mathrm{un}=0$$

This can be put in the form

$$(1-x^2)u_x``-2\mathrm{xu}_n`+n(n+1)\mathrm{un}=0---2$$

Comparing 2 with 1 we conclude that u_n is a solution of the legendr eq. it may be observed that U is a polynomial of degrees 2x & hence will be a polynomial of degree x.

also $p_x(x)$ which satisfies the legendre differentiate eq. is also polynomial of degree x.

$$p_n(x)=ku_n=k\left[(x^2-1)^n\right]_n$$

$$p_n(x)=k[(k-1)^n(x+1)^n]$$

Applying Leibnitz theorem for the RHS we have

$p_n(x)$

$$= k[(k-1)^n(x+1)^n]_n + n.n(x-1)^{n-1}[(x+1)^n]_{n-1} + \frac{n(n-1)}{2}n(n -$$

$$(x-1)^{n-2}[(x-1)^n]_{n-2} + ----- [(x-1)^n]_n$$

It should be observed that if $z = (x-1)^n$

$z_1 = n(x-1)^{n-1} \& z_2 = n(n-1)(x-1)^{n-2}$etc.

$z_n = n(n-1)(n-2)....2.1(x-1)^{n-1}$or

$z_n = n!(x-1)^{\circ}$

$z_n = n!$

$[(x-1)^n]_x, x!$

Putting x =1 in eq. 1 all the terms in RHS become zero except the l term which becomes $n!(1+1)^n = n!2^n$

$p_n(1) = 1$ by thedef of $p_n(x)$

$1 = \text{kn:}2^n$

$k = \frac{1}{n!2^n}$

$p_n(x) = ku_n$

$p_n(x) = \frac{1}{2^n n!}[(x^2-1)^n]$

$p_n(x) = \frac{1}{2^n n!}\frac{d^n}{d^x}(x^2-1)^n$

Q4. A coin is tossed. If it turns up H, two balls will be drawn from urn A otherwise 2 balls will be drawn from urn B. urn A contains 3 red and 5 blue balls , urn B contains 7 red and 5 blue balls. What is the probability that urn A is used , given that both balls and blue? (find in both cases, when balls were chosen with replacement and without replacement).

Sol- let us define the following events

$A_1 =$ urn A is chosen

$A_2 =$ urn B is chosen

E= two blue balls are drawn (with reputation)

Then we have

$p(A_1),\frac{1}{2}$

$p(E/A) = \frac{5}{8} * \frac{5}{8}$

$\frac{25}{64}$

$p(A_2),\frac{1}{2}$

$p(E/A) = \frac{5}{12} * \frac{5}{12}$

$\frac{25}{144}$

So,

$$p(A/E) = \frac{\left(P(A_1)\&P\left(E/A_1\right)\right)}{P(A_1)P\left(E/A_1\right) + P(A_2)P\left(E/A_2\right)}$$

$$\frac{\left(\frac{1}{2} * \frac{25}{64}\right)}{\frac{1}{2} * \frac{25}{64} + \frac{1}{2} * \frac{25}{144}}$$

$$\frac{25}{\frac{128}{25} + \frac{25}{288}}$$

$$\frac{\frac{25}{128}}{\frac{(225 + 100)}{1152}} = \frac{\frac{25}{128}}{\frac{325}{1152}}$$

$$\frac{9}{13}$$

(b) for event $A_1, A_2, A_3, A_4, A_5 \ldots A_n$

$$P\left(\cup^n i = 1 A_i\right) \geq \sum_{i=1}^{n} p(A_i) - (n-1)$$

Prove that

i. $p\left(\cap^n i = 1\right) \geq 1 - \sum_{i=1}^{n} p\left(\bar{A}_i\right)$

$$p\left(\cap^n i = 1 A_i\right) \geq 1 - \sum_{i=1}^{n} p\left(\bar{A}_i\right)$$

Q5. State and prove bayes theorem.

Sol. –it states that "If $A_1, A_2 - - - A_N$ are n mutually exclusive event with $P(A_1) \neq 0, i = 1,2 - - - n$ & B is any other event which can occurred with A or A_1 orA_N then we have,

$$P\left(A_i/B\right) = \frac{\left[P(A_i)P\left(B/A_i\right)\right]}{\sum_{i=1}^{n} p(A;)P\left(B/A;\right)}$$

Proof- by compound theorem of probability ,

We get

$$P(A;\cap B) = P(A_i).P\left(B/A;\right)$$

Or $P(A;\cap B) = P(B).P\left(A;/B\right)$

Given that , B is any other event which occur with A or A; or .....A_N i.e

$$B = B\cap(A_1 \text{or} A_i \text{or} - - - - A_n)$$

$$B\cap\left[A_1 \cup A_2 \bigcup - - - A_N\right]$$

$$(B\cap A_1)\cup(B\cap A_2) \bigcup - - - U(B\cap A_N)$$

$$P\left[B\cap A_1 \cup (B\cap A_2) \bigcup - - - - - P(B\cap A_n)\right]$$

$$p(B\cap A_1) + P(B\cap A_2) \pm - - P(B\cap A_n)$$

$$\sum_{i=1}^{n} p(B\cap A;)$$

$$\sum_{i=1}^{n} p(A;)P\left(B/A;\right)$$

Again from II

$$P(A_i/B) = \frac{[P(A_i)P(B/A_i)]}{\sum_{i=1}^{n} p(A_i)P(B/A_i)}$$

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(b) a random variable X follows binominal distribution with parameter n=40 and $p = \frac{1}{4}$ use chebyshev's inequality to find bounds for.

a. $p(|X - 10| < 8)$

b. $p(|X - 10| < 10)$